IMC 2024

Second Day, August 8, 2024

Problem 6. Prove that for any function $f\colon \mathbb{Q} \to \mathbb{Z}$, there exist $a, b, c \in \mathbb{Q}$ such that $a < b < c$, $f(b) \geq f(a)$, and $f(b) \geq f(c)$.

(10 points)

Problem 7. Let n be a positive integer. Suppose that A and B are invertible $n \times n$ matrices with complex entries such that $A + B = I$ (where I is the identity matrix) and

$$(A^2 + B^2)(A^4 + B^4) = A^5 + B^5.$$

Find all possible values of $\det(AB)$ for the given n.

(10 points)

Problem 8. Define the sequence $x_1, x_2, \ldots$ by the initial terms $x_1 = 2$, $x_2 = 4$, and the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for } n \geq 1.$$

Prove that $\lim\limits_{n\to\infty} \frac{x_n}{2^n}$ exists and satisfies

$$\frac{1+\sqrt{3}}{2} \leq \lim_{n\to\infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$

(10 points)

Problem 9. A matrix $A = (a_{ij})$ is called *nice*, if it has the following properties:

(i) the set of all entries of A is $\{1, 2, \ldots, 2t\}$ for some integer t;

(ii) the entries are non-decreasing in every row and in every column: $a_{i,j} \leq a_{i,j+1}$ and $a_{i,j} \leq a_{i+1,j}$;

(iii) equal entries can appear only in the same row or the same column: if $a_{i,j} = a_{k,\ell}$, then either $i = k$ or $j = \ell$;

(iv) for each $s = 1, 2, \ldots, 2t - 1$, there exist $i \neq k$ and $j \neq \ell$ such that $a_{i,j} = s$ and $a_{k,\ell} = s + 1$.

Prove that for any positive integers m and n, the number of nice $m \times n$ matrices is even.

For example, the only two nice 2×3 matrices are $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 4 \end{pmatrix}$.

(10 points)

Problem 10. We say that a square-free positive integer n is *almost prime* if

$$n \mid x^{d_1} + x^{d_2} + \ldots + x^{d_k} - kx$$

for all integers x, where $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n. Suppose that r is a Fermat prime (i.e. it is a prime of the form $2^{2^m} + 1$ for an integer $m \geq 0$), p is a prime divisor of an almost prime integer n, and $p \equiv 1 \pmod{r}$. Show that, with the above notation, $d_i \equiv 1 \pmod{r}$ for all $1 \leq i \leq k$.

(An integer n is called *square-free* if it is not divisible by d^2 for any integer $d > 1$.)

(10 points)

After the end of contest, the solutions and preliminary results will be posted at
https://imc-math.org.uk/?year=2024&item=problems and
https://imc-math.org.uk/?year=2024&item=results.