IMC 2024

First Day, August 7, 2024

Solutions

Problem 1. Determine all pairs $(a, b) \in \mathbb{C} \times \mathbb{C}$ satisfying

$$|a| = |b| = 1 \quad \text{and} \quad a + b + a\bar{b} \in \mathbb{R}.$$

(proposed by Mike Daas, Universiteit Leiden)

Hint: Write $a = e^{ix}$ and $b = e^{iy}$, and transform the RHS to a product.

Solution 1. Write $a = e^{ix}$ and $b = e^{iy}$ for some $x, y \in [0, 2\pi)$. Using Euler's formula, and the well-known identities

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \quad \text{and} \quad \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2},$$

we get a product form of the left-hand side as

$$\begin{aligned} \operatorname{Im}\left(a + b + a\bar{b}\right) &= \big(\sin x + \sin y\big) + \sin(x - y) \\ &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} + 2 \sin \frac{x-y}{2} \cos \frac{x-y}{2} \\ &= 2\left(\sin \frac{x+y}{2} + \sin \frac{x-y}{2}\right) \cos \frac{x-y}{2} \\ &= 4 \sin \frac{x}{2} \cdot \cos \frac{y}{2} \cdot \cos \frac{x-y}{2}. \end{aligned}$$

Hence, $a + b + a\bar{b}$ is real if and only if either $\sin \frac{x}{2} = 0$, $\cos \frac{y}{2} = 0$ or $\cos \frac{x-y}{2} = 0$, which respectivelly correspond to $x = 2k\pi$, $y = (2k+1)\pi$ and $x = y + (2k+1)\pi$.

Therefore, the solutions are

$$(1, b), \quad (a, -1) \quad \text{and} \quad (a, -a) \quad \text{with } |a| = 1,\ |b| = 1.$$

Solution 2. Notice that

$$a + b + a\bar{b} \in \mathbb{R} \iff 1 + a + b + a\bar{b} \in \mathbb{R}.$$

Let $c \in \mathbb{C}$ be such that $a = c^2$. Now observe that

$$\begin{aligned} \bar{c}(1 + a + b + a\bar{b}) &= \bar{c} + \bar{c}c^2 + \bar{c}b + \bar{c}c^2\bar{b} \\ &= \bar{c} + c + \bar{c}b + c\bar{b} \in \mathbb{R}, \end{aligned}$$

where we used that $\bar{c}c = 1$ and $z + \bar{z} \in \mathbb{R}$ for any $z \in \mathbb{C}$. W e conclude that either $c \in \mathbb{R}$, or $1 + a + b + a\bar{b} = 0$. In the first case, $c = \pm 1$ and so $a = 1$. In the second case, we factor the equation as

$$(a + b)(1 + \bar{b}) = 1 + a + 1b + a\bar{b} = 0, \quad \text{and as such,} \quad a = -b \quad \text{or} \quad b = -1.$$

We find precisely three families of pairs (a, b): the pairs $(1, b)$ for b on the unit circle; the pairs $(a, -1)$ for a on the unit circle; and the pairs $(a, -a)$ for a on the unit circle.

Problem 2. For $n = 1, 2, \ldots$ let

$$S_n = \log\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \ldots \cdot n^n}\right) - \log(\sqrt{n}),$$

where log denotes the natural logarithm. Find $\lim\limits_{n\to\infty} S_n$.

(proposed by Sergey Chernov, Belarusian State University, Minsk)

Hint: S_n is (close to) a Riemann sum of a certain integral.

Solution. Transform S_n as

$$\begin{aligned}
S_n &= \frac{1}{n^2}\sum_{k=1}^{n} k\log k - \frac{1}{2}\log n \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\left(\log\frac{k}{n} + \log n\right)\right) - \frac{1}{2}\log n \\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{n^2}\sum_{k=1}^{n} k - \frac{1}{2}\log n \\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{2n}.
\end{aligned}$$

Here the last term $\dfrac{\log n}{2n}$ converges to 0. The sum $\dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{k}{n}\log\dfrac{k}{n}$ is a Riemann sum for the integrable function $f(x) = x\log x$ on the segment $[0,1]$ with the uniform grid $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Therefore

$$\lim \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} = \lim\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 x\log x\,\mathrm{d}x = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1 = -\frac{1}{4}.$$

Hence, $\lim S_n$ exists, and $\lim S_n = -\dfrac{1}{4}$.

Problem 3. For which positive integers n does there exist an $n \times n$ matrix A whose entries are all in $\{0, 1\}$, such that A^2 is the matrix of all ones?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Let J be the $n \times n$ matrix with all ones. Consider $A^3 = AJ = JA$.

Solution. Answer: Such a matrix A exists if and only if n is a complete square.

Let J_n be the $n \times n$ matrix with all ones, so $A^2 = J_n$. Consider the equality

$$A^3 = AJ_n = J_nA.$$

In the matrix AJ_n, all columns are equal to the sum of colums in A, that is, the (i, j)th entry in AJ_n is the number of ones in the ith row of A. Similarly, the (i, j)th entry in J_nA is the number of ones in the jth column of A. These numbers must be equal, so A contains the same number of ones in every row and every column. Let this common number be k; then $AJ_n = J_nA = kJ_n$.

Now from

$$nJ_n = J_n^2 = (A^2)^2 = A(AJ_n) = A(kJ_n) = k(AJ_n) = k^2 J_n$$

we can read $n = k^2$, so n must be a complete square.

It remains to show an example for a matrix A of order $n = k^2$. For $l = 0, 1, \dots, k-1$, let B_l be the $k \times k$ matrix whose (i, j)th entry is 1 if $j - i \equiv l \pmod{k}$ and 0 otherwise, i.e., B_l can be obtained from the identity matrix by cyclically shifting the colums l times, and let

$$A = \begin{pmatrix} B_0 & B_1 & B_2 & \dots & B_{k-1} \\ B_0 & B_1 & B_2 & \dots & B_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_0 & B_1 & B_2 & \dots & B_{k-1} \end{pmatrix};$$

The (i, j)th block in A^2 is

$$\begin{pmatrix} B_0 & B_1 & \dots & B_{k-1} \end{pmatrix} \begin{pmatrix} B_{j-1} \\ \vdots \\ B_{j-1} \end{pmatrix} = (B_0 + B_1 + \dots + B_{k-1})B_{j-1} = J_k B_{j-1} = J_k,$$

so this matrix indeed satisfies $A^2 = J_{k^2}$.

Problem 4. Let g and h be two distinct elements of a group G, and let n be a positive integer. Consider a sequence $w = (w_1, w_2, \dots)$ which is not eventually periodic and where each w_i is either g or h. Denote by H the subgroup of G generated by all elements of the form $w_k w_{k+1} \dots w_{k+n-1}$ with $k \geq 1$. Prove that H does not depend on the choice of the sequence w (but may depend on n).

(proposed by Ivan Mitrofanov, Saarland University)

Solution. Let X_m denote the subset of G of products of the form $g_1 \dots g_m$, where each g_i is either g or h.

Lemma. For all $j = 1, 2, \dots, n$ and for all $a, b \in X_j$ the ratio $a^{-1}b$ is contained in H.

Proof. Induction in j.

We start with the base case $j = 1$. By the pigeonhole principle, there exist $k < \ell$ for which the sequences $(w_{k+1}, \dots, w_{k+n-1})$ and $(w_{\ell+1}, \dots, w_{\ell+n-1})$ coincide. If $w_{k+m} = w_{\ell+m}$ for all positive integer m, then the sequence w is eventually periodic with period $\ell - k$. Thus, there exists $m > 0$ for which $w_{k+m} \neq w_{\ell+m}$. We have $m \geqslant n$, so $w_{k+m-i} = w_{\ell+m-i}$ for $i = 1, 2, \dots, n-1$. Therefore, since the products $x = w_{k+m-n+1} \dots w_{k+m}$ and $y = w_{\ell+m-n+1} \dots w_{\ell+m}$ both are elements of H, the subgroup H contains their ratios $x^{-1}y$ and $y^{-1}x$. These ratios are equal to $g^{-1}h$ and $h^{-1}g$ (in some order), that finishes the proof for $j = 1$.

Induction step from $j-1$ to j, $2 \leqslant j \leqslant n$. We say that an element $a \in X_j$ is a g-element, correspondingly an h-element, if it can be represented as $a = ga_1$, correspondingly $a = ha_1$, where $a_1 \in X_{j-1}$. The ratio of two g-elements, or of two h-elements, is a ratio of two elements of X_{j-1}, thus, it is in H by the induction hypothesis. Since the property $a^{-1}b \in H$ is an equivalence relation on pairs (a, b), it suffices to find a g-element and h-element whose ratio is in H.

Define k, ℓ, m, as in the base case. The subgroup H contains the products

$$v = w_{k+m-n+j} \dots w_{k+m} w_{k+m+1} \dots w_{k+m+j-1},$$
$$u = w_{\ell+m-n+j} \dots w_{\ell+m} w_{\ell+m+1} \dots w_{\ell+m+j-1}.$$

Their ratio $u^{-1}v$ is a ratio of g-element and an h-element in X_j, since $\{w_{k+m}, w_{\ell+m}\} = \{g, h\}$ and $w_{k+m-i} = w_{\ell+m-i}$ for all $i = 1, 2, \dots, n-j$.

The Lemma for $j = n$ yields that H is the subgroup of G generated by X_n, and this description does not depend on w.

Problem 5. Let $n > d$ be positive integers. Choose n independent, uniformly distributed random points $x_1, \ldots, x_n$ in the unit ball $B \subset \mathbb{R}^d$ centered at the origin. For a point $p \in B$ denote by $f(p)$ the probability that the convex hull of $x_1, \ldots, x_n$ contains p. Prove that if $p, q \in B$ and the distance of p from the origin is smaller than the distance of q from the origin, then $f(p) \geq f(q)$.

(proposed by Fedor Petrov, St Petersburg State University)

Solution. By radial symmetry of the distribution, $f(p)$ depends only on $|op|$ (the distance between o and p), so, we may assume that p lies on the segment between o and q. For points $x_1, \ldots, x_n$ and $x \in B$ denote by $f_x(x_1, \ldots, x_n)$ the indicator function of the event "x is in the convex hull of $x_1, \ldots, x_n$". The claim follows from the following deterministic inequality

$$\sum f_p(\pm x_1, \ldots, \pm x_n) \geqslant \sum f_q(\pm x_1, \ldots, \pm x_n), \tag{1}$$

where $x_1, \ldots, x_n \in B$ are arbitrary points in general position and the summations are over all 2^n choices of signs (here o is identified with the origin, that is, x and $-x$ are symmetric with respect to o). Indeed, taking the expectation in (1) over independent random uniform $x_1, \ldots, x_n$, we get $2^n f(p) \geqslant 2^n f(q)$. (To be specific, here "general position" means that for any point set $A \subset \{\pm x_1, \ldots, \pm x_n, p, q\}$, which does not contain simultaneosuly x_i and $-x_i$, is not contained in an (affine) $(|A|-2)$-dimensional plane. This holds with probability 1.)

To prove (1), we use the following formula for the characteristic function χ_P of the convex polyhedron $P \subset \mathbb{R}^d$: if $P_1, \ldots, P_k$ are all facets of P, and Q_i is the convex hull of o and P_i, then $\chi_P = \sum \pm \chi_{Q_i}$, where the sign is plus if o and P are on the same side of P_i, and minus otherwise. Indeed, for every point p in general position look how the ray op intersects the boundary of P and realize that for at most two summands the contribution of the RHS at point p is non-zero, and the total contribution equals 1 when p is inside P and 0 (possibly as $0 = 1 - 1$) otherwise. Use this formula for every polyhedron P with n vertices $y_1, \ldots, y_n$, where each y_i is $\pm x_i$. These polyhedrons are simplicial (all facets are simplices) because of the general position condition. Sum up over all 2^n such P, we get the expression of $\sum_P \chi_P$ as a linear combination of χ_S, where S are simplices formed by o and some d points in $\{\pm x_1, \ldots, \pm x_n\}$ (not containing x_i and $-x_i$ simultaneously).

For proving (1), it suffices to verify that all coefficients of χ_S in this linear combination are positive (since two sides of (1) are the values of the sum $\sum_P \chi_P$ at p and q). Let's find a coefficient of χ_S, where, say, S is a simplex with vertices $o, x_1, \ldots, x_d$. The plane α through $x_1, \ldots, x_d$ partitions $\mathbb{R}^d$ onto two parts H^+ (containing o) and H^- (not containing o). For every pair $\{x_i, -x_i\}$ with $i > d$, either both points belong to H^+, or one belongs to H^- and another to H^+. χ_S goes with the plus sign for P with vertices $x_1, \ldots, x_d$ and other vertices from H^+, and with the minus sign for P with vertices $x_1, \ldots, x_d$ and other vertices from H^-. It is immediate that there are at least as many pluses as minuses.