IMC 2023

First Day, August 2, 2023

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

(10 points)

Problem 2. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \quad \text{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

(10 points)

Problem 3. Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x,y)P(z,t) = P(xz - yt, xt + yz).$$

(10 points)

Problem 4. Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ form a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?

(10 points)

Problem 5. Fix positive integers n and k such that $2 \le k \le n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He realized that for every preferred permutation x, there exist k indices $i_1 < i_2 < \ldots < i_k$ with the following property: for every $1 \le j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$, he obtains another preferred permutation.

Prove that he prefers at least $k!$ permutations.

(10 points)