IMC 2022

First Day, August 3, 2022

Problem 1. Let $f : [0,1] \to (0,\infty)$ be an integrable function such that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\ \mathrm{d}x \geq 1.$$

(10 points)

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.

(A^T denotes the transpose of A.)

(10 points)

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

(10 points)

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \leq a < b < c < d \leq n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log \log n \leqslant m \leqslant 100 \log \log n.$$

(10 points)