IMC 2021 Online

First Day, August 3, 2021

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(10 points)

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \ldots, k+a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \ldots, k+n+a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

(10 points)

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \ldots \in (0, d)$ such that for each n, the points $a_1, \ldots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find

$$\sup\Big\{d \;\Big|\; d \text{ is good}\Big\}.$$

(10 points)

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x - y| < \min\big\{g(x), g(y)\big\}, \quad \text{then} \quad \big|f(x) - f(y)\big| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

(10 points)