IMC 2017, Blagoevgrad, Bulgaria

Day 2, August 3, 2017

Problem 6. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x \to +\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n \to \infty} \int_0^1 f(nx)\, \mathrm{d}x = L.$$

(10 points)

Problem 7. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

(10 points)

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

(10 points)

Problem 9. Define the sequence $f_1, f_2, \ldots : [0, 1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0,1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim\limits_{n \to \infty} f_n(x)$ exists for every $x \in [0, 1)$ and determine the limit function.

(10 points)

Problem 10. Let K be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If n is a positive integer, and $T_1, \ldots, T_n$ are non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{\ell=1}^{n} \operatorname{area}(T_\ell) > \operatorname{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \operatorname{perimeter}(T_\ell) > p.$$

(10 points)