IMC 2017, Blagoevgrad, Bulgaria

Day 1, August 2, 2017

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. By taking squares,

$$A^4 = (A^2)^2 = (A^T)^2 = (A^2)^T = (A^T)^T = A,$$

so

$$A^4 - A = 0;$$

it follows that all eigenvalues of A are roots of the polynomial $X^4 - X$.

The roots of $X^4 - X = X(X^3 - 1)$ are 0, 1 and $\frac{-1 \pm \sqrt{3}i}{2}$. In order to verify that these values are possible, consider the matrices

$$A_0 = \begin{pmatrix} 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The numbers 0 and 1 are the eigenvalues of the 1×1 matrices A_0 and A_1, respectively. The numbers $\frac{-1 \pm \sqrt{3}i}{2}$ are the eigenvalues of A_2; it is easy to check that

$$A_2^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = A_2^T.$$

The matrix A_4 establishes all the four possible eigenvalues in a single matrix.

Remark. The matrix A_2 represents a rotation by $2\pi/3$.

Problem 2. Let $f\colon \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$\left| f'(x) - f'(y) \right| \le L \left| x - y \right|$$

for all x, y. Prove that

$$\left(f'(x) \right)^2 < 2L f(x)$$

holds for all x.

(Proposed by Jan Šustek, University of Ostrava)

Solution. Notice that f' satisfies the Lipschitz-property, so f' is continuous and therefore locally integrable.

Consider an arbitrary $x \in \mathbb{R}$ and let $d = f'(x)$. We need to prove $f(x) > \frac{d^2}{2L}$.

If $d = 0$ then the statement is trivial.

If $d > 0$ then the condition provides $f'(x-t) \geq d - Lt$; this estimate is positive for $0 \leq t < \frac{d}{L}$. By integrating over that interval,

$$f(x) > f(x) - f(x - \tfrac{d}{L}) = \int_0^{\frac{d}{L}} f'\big(x - t\big)\mathrm{d}t \geq \int_0^{\frac{d}{L}} (d - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

If $d < 0$ then apply $f'(x+t) \leq d + Lt = -|d| + Lt$ and repeat the same argument as

$$f(x) > f(x) - f(x + \tfrac{|d|}{L}) = \int_0^{\frac{|d|}{L}} \big(-f'(x+t)\big)\mathrm{d}t \geq \int_0^{\frac{|d|}{L}} (|d| - Lt)\mathrm{d}t = \frac{d^2}{2L}.$$

Problem 3. For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016).$$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \leq k \leq 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

(Proposed by Matko Ljulj , University of Zagreb)

Solution. We prove that the answer is yes; for every $S \subset \{1, 2, \ldots, 2017\}$ there exists a suitable n. Specially, n can be a power of 2: $n = 2^{w_1}$ with some nonnegative integer w_1. Write $a_k(n) = 2^{w_k}$; then

$$2^{w_{k+1}} = a_{k+1}(n) = P(a_k(n)) = P(2^{w_k}) = 1 \cdot 2 \cdot 4 \cdots 2^{w_k} = 2^{\frac{w_k(w_k+1)}{2}},$$

so

$$w_{k+1} = \frac{w_k(w_k+1)}{2}.$$

The proof will be completed if we prove that for each choice of S there exists an initial value w_1 such that w_k is even if and only if $k \in S$.

Lemma. Suppose that the sequences $(b_1, b_2, \ldots)$ and $(c_1, c_2, \ldots)$ satisfy $b_{k+1} = \frac{b_k(b_k+1)}{2}$ and $c_{k+1} = \frac{c_k(c_k+1)}{2}$ for $k \geq 1$, and $c_1 = b_1 + 2^m$. Then for each $k = 1, \ldots m$ we have $c_k \equiv b_k + 2^{m-k+1} \pmod{2^{m-k+2}}$.

As an immediate corollary, we have $b_k \equiv c_k \pmod 2$ for $1 \leq k \leq m$ and $b_{m+1} \equiv c_{m+1} + 1 \pmod 2$.

Proof. We prove the by induction. For $k = 1$ we have $c_1 = b_1 + 2^m$ so the statement holds. Suppose the statement is true for some $k < m$, then for $k+1$ we have

$$\begin{aligned} c_{k+1} &= \frac{c_k\,(c_k+1)}{2} \equiv \frac{\big(b_k + 2^{m-k+1}\big)\big(b_k + 2^{m-k+1} + 1\big)}{2} \\ &= \frac{b_k^2 + 2^{m-k+2}b_k + 2^{2m-2k+2} + b_k + 2^{m-k+1}}{2} = \\ &= \frac{b_k(b_k+1)}{2} + 2^{m-k} + 2^{m-k+1}b_k + 2^{2m-2k+1} \equiv \frac{b_k(b_k+1)}{2} + 2^{m-k} \pmod{2^{m-k+1}}, \end{aligned}$$

therefore $c_{k+1} \equiv b_{k+1} + 2^{m-(k+1)+1} \pmod{2^{m-(k+1)+2}}$.

Going back to the solution of the problem, for every $1 \le m \le 2017$ we construct inductively a sequence $(v_1, v_2, \ldots)$ such that $v_{k+1} = \frac{v_k(v_k+1)}{2}$, and for every $1 \le k \le m$, v_k is even if and only if $k \in S$.

For $m = 1$ we can choose $v_1 = 0$ if $1 \in S$ or $v_1 = 1$ if $1 \notin S$. If we already have such a sequence $(v_1, v_2, \ldots)$ for a positive integer m, we can choose either the same sequence or choose $v_1' = v_1 + 2^m$ and apply the same recurrence $v_{k+1}' = \frac{v_k'(v_k'+1)}{2}$. By the Lemma, we have $v_k \equiv v_k' \pmod 2$ for $k \le m$, but v_{m+1} and v_{m+1} have opposite parities; hence, either the sequence (v_k) or the sequence (v_k') satisfies the condition for $m + 1$.

Repeating this process for $m = 1, 2, \ldots, 2017$, we obtain a suitable sequence (w_k).

Problem 4. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $n/2017$ persons in S have exactly two friends in S.

(Proposed by Rooholah Majdodin and Fedor Petrov, St. Petersburg State University)

Solution. Let $d = 1000$ and let $0 < p < 1$. Choose the set S randomly such that each people is selected with probability p, independently from the others.

The probability that a certain person is selected for S and knows exactly two members of S is

$$q = \binom{d}{2} p^3 (1-p)^{d-2}.$$

Choose $p = 3/(d+1)$ (this is the value of p for which q is maximal); then

$$q = \binom{d}{2} \left(\frac{3}{d+1}\right)^3 \left(\frac{d-2}{d+1}\right)^{d-2} =$$

$$= \frac{27d(d-1)}{2(d+1)^3} \left(1 + \frac{3}{d-2}\right)^{-(d-2)} > \frac{27d(d-1)}{2(d+1)^3} \cdot e^{-3} > \frac{1}{2017}.$$

Hence, $E\big(|S|\big) = nq > \frac{n}{2017}$, so there is a choice for S when $|S| > \frac{n}{2017}$.

Problem 5. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2} z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0 c_{n-2} = c_1 c_{n-3} = \ldots = c_{n-2} c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

(Proposed by Vsevolod Lev and Fedor Petrov, St. Petersburg State University)

Solution. Let $M = \{z : z^n = 1\}$, $A = \{z \in M : f(z) \neq 0\}$ and $A^{-1} = \{z^{-1} : z \in A\}$. We have to prove $|A| \ge k$.

Claim.

$$A \cdot A^{-1} = M.$$

That is, for any $\eta \in M$, there exist some elements $a, b \in A$ such that $ab^{-1} = \eta$.

Proof. As is well-known, for every integer m,

$$\sum_{z \in M} z^m = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

Define $c_{n-1}=1$ and consider

$$\sum_{z\in M} z^2 f(z)f(\eta z) = \sum_{z\in M} z^2 \sum_{j=0}^{n-1} c_j z^j \sum_{\ell=0}^{n-1} c_\ell (\eta z)^\ell = \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z\in M} z^{j+\ell+2} =$$

$$= \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z\in M} \begin{Bmatrix} n & \text{if } n|j+\ell+2 \\ 0 & \text{otherwise} \end{Bmatrix} = c_{n-1}^2 n + \sum_{j=0}^{n-2} c_j c_{n-2-j} \eta^{n-2-j} n = n \neq 0.$$

Therefore there exists some $b\in M$ such that $f(b)\neq 0$ and $f(\eta b)\neq 0$, i.e. $b\in A$, and $a=\eta b\in A$, satisfying $ab^{-1}=\eta$.

By double-counting the elements of M, from the Claim we conclude

$$|A|\big(|A|-1\big) \geq \big|M\setminus\{1\}\big| = n-1 \geq k^2-3k+3 > (k-1)(k-2)$$

which shows $|A|>k-1$.