IMC 2014, Blagoevgrad, Bulgaria

Day 2, August 1, 2014

Problem 1. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \dots, 9\}$ and $x = \sum\limits_{n=1}^{\infty} d_n(x)10^{n-1}$. Suppose that for some sequence $\left(a_n\right)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $\left(d_n(a_n)\right)_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution 1. By the assumption there is some index n_0 such that $d_n(a_n) \neq 1$ for $n \geq n_0$. We show that

$$a_{n+1}, a_{n+2}, \ldots > 10^n \quad \text{for} \quad n \geq n_0. \tag{1}$$

Notice that in the sum $a_n = \sum\limits_{k=1}^{\infty} d_k(a_n)10^{k-1}$ we have the term $d_n(a_n)10^{n-1}$ with $d_n(a_n) \geq 1$. Therefore, $a_n \geq 10^{n-1}$. Then for $m > n$ we have $a_m \geq 10^m > 10^n$. This proves (1).

From (1) we know that only the first n elements, $a_1, a_2, \ldots, a_n$ may lie in the interval $[1, 10^n]$. Hence, at least $10^n - n$ integers in this interval do not occur in the sequence at all. As $\lim(10^n - n) = \infty$, this shows that there are infinitely many numbers that do not appear among $a_1, a_2, \ldots$.

Solution 2. We will use Cantor's diagonal method to construct infinitely many positive integers that do not occur in the sequence (a_n)

Assume that $d_n(a_n) \neq 0$ for $n > n_0$. Define the sequence of digits

$$g_n = \begin{cases} 2 & d_n(x_n) = 1 \\ 1 & d_n(x_n) \neq 1. \end{cases}$$

Hence $g_n \neq d_n(a_n)$ for every positive integer n. Let

$$x_k = \sum_{n=1}^{k} g_n \cdot 10^{n-1} \quad \text{for} \quad k = 1, 2, \ldots.$$

As $x_{k+1} \geq 10^k > x_k$, the sequence (x_k) is increasing and so it contains infinitely many distinct positive integers. We show that the numbers $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ no not occur in the sequence (a_n); in other words, $x_k \neq a_n$ for every pair $n \geq 1$ and $k \geq n_0$ of integers.

Indeed, if $k \geq n$ then $d_n(x_k) = g_n \neq d_n(a_n)$, so $x_k \neq a_n$.

If $n > k \geq n_0$ then $d_n(x_k) = 0 \neq d_n(a_n)$, so $x_k \neq a_n$.

Problem 2. Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii}a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

(Proposed by Martin Niepel, Comenius University, Bratislava)

Solution. Eigenvalues of a real symmetric matrix are real, hence the inequality makes sense. Similarly, for Hermitian matrices diagonal entries as well as eigenvalues have to be real.

Since the trace of a matrix is the sum of its eigenvalues, for A we have

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

and consequently

$$\sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ii}a_{jj} = \sum_{i=1}^{n} \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j.$$

Therefore our inequality is equivalent to

$$\sum_{i=1}^{n} a_{ii}^2 \le \sum_{i=1}^{n} \lambda_i^2.$$

Matrix A^2, which is equal to A^TA (or A^*A in Hermitian case), has eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$. On the other hand, the trace of A^TA gives the square of the Frobenius norm of A, so we have

$$\sum_{i=1}^{n} a_{ii}^2 \le \sum_{i,j=1}^{n} |a_{ij}|^2 = \operatorname{tr}(A^TA) = \operatorname{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2.$$

The inequality follows, and it is clear that the equality holds for diagonal matrices only.

Remark. Same statement is true for Hermitian matrices.

Problem 3. Let $f(x) = \dfrac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $\left|f^{(n)}(x)\right| < \dfrac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution 1. Putting $f(0) = 1$ we can assume that the function is analytic in $\mathbb{R}$. Let $g(x) = x^{n+1}(f^n(x) - \frac{1}{n+1})$. Then $g(0) = 0$ and

$$g'(x) = (n+1)x^n \left(f^{(n)}(x) - \frac{1}{n+1} \right) + x^{n+1} f^{(n+1)}(x) =$$

$$= x^n \left((n+1)f^{(n)}(x) + xf^{(n+1)}(x) - 1 \right) = x^n \left((xf(x))^{(n+1} - 1 \right) = x^n(\sin^{(n+1)}(x) - 1) \le 0.$$

Hence $g(x) \le 0$ for $x > 0$. Taking into account that $g'(x) < 0$ for $0 < x < \frac{\pi}{2}$ we obtain the desired (strict) inequality for $x > 0$.

Solution 2.

$$\left(\frac{\sin x}{x}\right)^{(n)}=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^1-\cos(xt)\mathrm{d}t=\int_0^1\frac{\partial^n}{\partial x^n}\left(-\cos(xt)\right)\mathrm{d}t=\int_0^1 t^n g_n(xt)\mathrm{d}t$$

where the function $g_n(u)$ can be $\pm\sin u$ or $\pm\cos u$, depending on n. We only need that $|g_n|\le 1$ and equality occurs at finitely many points. So,

$$\left|\left(\frac{\sin x}{x}\right)^{(n)}\right|\le\int_0^1 t^n\big|g_n(xt)\big|\mathrm{d}t<\int_0^1 t^n\mathrm{d}t=\frac{1}{n+1}.$$

Problem 4. We say that a subset of $\mathbb{R}^n$ is k-*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers k and n, find the minimal number $d(k,n)$ such that every finite k-generic set in $\mathbb{R}^n$ contains a k-generic subset with at most $d(k,n)$ elements.
(Proposed by Shachar Carmeli, Weizmann Inst. and Lev Radzivilovsky, Tel Aviv Univ.)

Solution. The answer is: $d(k,n)=\begin{cases}k\cdot n & k,n>1\\ k+n & \text{otherwise}\end{cases}$

Throughout the solution, we shall often say that a hyperplanes **skips** a point to signify that the plane does not contain that point.

For $n=1$ the claim is obvious.

For $k=1$ we have an arbitrary finite set of points in $\mathbb{R}^n$ such that neither hyperplane contains it entirely. We can build a subset of $n+1$ points step by step: on each step we add a point, not contained in the minimal plane spanned by the previous points. Thus any 1-generic set contains a non-degenerate simplex of $n+1$ points, and obviously a non-degenerate simplex of $n+1$ points cannot be reduced without loosing 1-generality.

In the case $k,n>1$ we shall give an example of $k\cdot n$ points. On each of the Cartesian axes choose k distinct points, different from the origin. Let's show that this set is k-generic. There are two types of planes: containing the origin and skipping it. If a plane contains the origin, it either contains all the chose points of a axis or skips all of them. Since no plane contains all axes, it skips the k chosen points on one of the axes. If a plane skips the origin, it it contains at most one point of each axis. Therefore it skips at least $n(k-1)$ points. It remains to verify a simple inequality $n(k-1)\ge k$ which is equivalent to $(n-1)(k-1)\ge 1$ which holds for $n,k>1$.

The example we have shown is minimal by inclusion: if any point is removed, say a point from axis i, then the hyperplane $x_i=0$ skips only $k-1$ points, and our set stops being k-generic. Hence $d(k,n)\ge kn$.

It remains to prove that Hence $d(k,n)\ge kn$ for $k,n>1$, meaning: for each k-generic finite set of points, it is possible to choose a k-generic subset of at most kn points. Let us call a subset of points **minimal** if by taking out any point, we loose k-generality. It suffices to prove that any minimal k-generic subset in $\mathbb{R}^n$ has at most kn points. A hyperplane will be called **ample** if it skips precisely k points. A point cannot be removed from a k-generic set, if and only if it is skipped by an ample hyperplane. Thus, in a minimal set each point is skipped by an ample hyperplane.

Organise the following process: on each step we choose an ample hyperplane, and paint blue all the points which are skipped by it. Each time we choose an ample hyperplane, which skips one of the unpainted points. The unpainted points at each step (after the beginning) is the intersection of all chosen hyperplanes. The intersection set of chosen hyperplanes is reduced with each step (since at least one point is being painted on each step).

Notice, that on each step we paint at most k points. So if we start with a minimal set of more then nk points, we can choose n planes and still have at least one unpainted points. The intersection of the chosen planes is a point (since on each step the dimension of the intersection plane was reduced), so there are at most $nk+1$ points in the set. The last unpainted point will be denoted by O. The last unpainted line (which was formed on the step before the last) will be denoted by ℓ_1.

This line is an intersection of all the chosen hyperplanes except the last one. If we have more than nk points, then ℓ_1 contains exactly $k+1$ points from the set, one of which is O.

We could have executed the same process with choosing the same hyperplanes, but in different order. Anyway, at each step we would paint at most k points, and after n steps only O would remain unpainted; so it was precisely k points on each step. On step before the last, we might get a different line, which is intersection of all planes except the last one. The lines obtained in this way will be denoted $\ell_1, \ell_2, ..., \ell_n$, and each contains exactly k points except O. Since we have O and k points on n lines, that is the entire set. Notice that the vectors spanning these lines are linearly independent (since for each line we have a hyperplane containing all the other lines except that line). So by removing O we obtain the example that we've described already, which is k-generic.

Remark. From the proof we see, that the example is unique.

Problem 5. For every positive integer n, denote by D_n the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_j \neq j$ for every $1 \leq j \leq n$. For $1 \leq k \leq \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_i = k + i$ for every $1 \leq i \leq k$ and $x_j \neq j$ for every $1 \leq j \leq n$. Prove that

$$\Delta(n,k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}.$$

(Proposed by Combinatorics; Ferdowsi University of Mashhad, Iran; Mirzavaziri)

Solution. Let $a_r \in \{i_1, \dots, i_k\} \cap \{a_1, \dots, a_k\}$. Thus $a_r = i_s$ for some $s \neq r$. Now there are two cases:

Case 1. $a_s \in \{i_1, \dots, i_k\}$. Let $a_s = i_t$. In this case a derangement $x = (x_1, \dots, x_n)$ satisfies the condition $x_{i_j} = a_j$ if and only if the derangement $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$ of the set $[n] \setminus \{i_t\}$ satisfies the condition $x'_{i_j} = a'_j$ for all $j \neq t$, where $a'_j = a_j$ for $j \neq s$ and $a'_s = a_t$. This provides a one to one correspondence between the derangements $x = (x_1, \dots, x_n)$ of $[n]$ with $x_{i_j} = a_j$ for the given sets $\{i_1, \dots, i_k\}$ and $\{a_1, \dots, a_k\}$ with ℓ elements in their intersections, and the derangements $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$ of $[n] \setminus \{i_t\}$ with $x_{i_j} = a'_j$ for the given sets $\{i_1, \dots, i_k\} \setminus \{i_t\}$ and $\{a'_1, \dots, a'_k\} \setminus \{a'_t\}$ with $\ell - 1$ elements in their intersections.

Case 2. $a_s \notin \{i_1, \dots, i_k\}$. In this case a derangement $x = (x_1, \dots, x_n)$ satisfies the condition $x_{i_j} = a_j$ if and only if the derangement $x' = (x'_1, \dots, x'_{a_s-1}, x'_{a_s+1}, x'_n)$ of the set $[n]\setminus\{a_s\}$ satisfies the condition $x'_{i_j}=a_j$ for all $j\neq s$. This provides a one to one correspondence between the derangements $x=(x_1,\dots,x_n)$ of $[n]$ with $x_{i_j}=a_j$ for the given sets $\{i_1,\dots,i_k\}$ and $\{a_1,\dots,a_k\}$ with ℓ elements in their intersections, and the derangements $x'=(x'_1,\dots,x'_{a_s-1},x'_{a_s+1},x'_n)$ of $[n]\setminus\{a_s\}$ with $x_{i_j}=a_j$ for the given sets $\{i_1,\dots,i_k\}\setminus\{i_s\}$ and $\{a_1,\dots,a_k\}\setminus\{a_s\}$ with $\ell-1$ elements in their intersections.

These considerations show that $\Delta(n,k,\ell)=\Delta(n-1,k-1,\ell-1)$. Iterating this argument we have

$$\Delta(n,k,\ell)=\Delta(n-\ell,k-\ell,0).$$

We can therefore assume that $\ell=0$. We thus evaluate $\Delta(n,k,0)$, where $2k\leqslant n$. For $k=0$, we obviously have $\Delta(n,0,0)=D_n$. For $k\geqslant 1$, we claim that

$$\Delta(n,k,0)=\Delta(n-1,k-1,0)+\Delta(n-2,k-1,0).$$

For a derangement $x=(x_1,\dots,x_n)$ satisfying $x_{i_j}=a_j$ there are two cases: $x_{a_1}=i_1$ or $x_{a_1}\neq i_1$.

If the first case occurs then we have to evaluate the number of derangements of the set $[n]\setminus\{i_1,a_1\}$ for the given sets $\{i_2,\dots,i_k\}$ and $\{a_2,\dots,a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-2,k-1,0)$.

If the second case occurs then we have to evaluate the number of derangements of the set $[n]\setminus\{a_1\}$ for the given sets $\{i_2,\dots,i_k\}$ and $\{a_2,\dots,a_k\}$ with 0 elements in their intersections. The number is equal to $\Delta(n-1,k-1,0)$.

We now use induction on k to show that

$$\Delta(n,k,0)=\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)},\quad 2\leqslant 2k\leqslant n.$$

For $k=1$ we have

$$\Delta(n,1,0)=\Delta(n-1,0,0)+\Delta(n-2,0,0)=D_{n-1}+D_{n-2}=\frac{D_n}{n-1}.$$

Now let the result be true for $k-1$. We can write

$$\begin{aligned}
\Delta(n,k,0) &= \Delta(n-1,k-1,0)+\Delta(n-2,k-1,0)\\
&= \sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{n-(k-1+i)}}{(n-1)-(k-1+i)}+\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n-1)-(k-1+i)}}{(n-2)-(k-1+i)}\\
&= \sum_{i=0}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\sum_{i=1}^{k-1}\binom{k-2}{i-1}\frac{D_{n-(k+i-1)}}{(n-1)-(k+i-1)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-2}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}\\
&\quad+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}+\sum_{i=1}^{k-2}\binom{k-2}{i-1}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}[\binom{k-2}{i}+\binom{k-2}{i-1}]\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\
&= \frac{D_{(n+1)-k}}{n-k}+\sum_{i=1}^{k-2}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}+\frac{D_{(n+1)-(2k-1)}}{n-(2k-1)}\\
&= \sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{(n+1)-(k+i)}}{n-(k+i)}.
\end{aligned}$$

Remark. As a corollary of the above problem, we can solve the first problem. Let $n=2k$, $i_j=j$ and $a_j=k+j$ for $j=1,\dots,k$. Then a derangement $x=(x_1,\dots,x_n)$ satisfies the condition $x_{i_j}=a_j$ if and only if $x'=(x_{k+1},\dots,x_n)$ is a permutation of $[k]$. The number of such permutations x' is $k!$. Thus $\sum_{i=0}^{k-1}\binom{k-1}{i}\frac{D_{k+1-i}}{k-i}=k!$.