IMC 2014, Blagoevgrad, Bulgaria

Day 2, August 1, 2014

Problem 1. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \dots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x) 10^{n-1}$. Suppose that for some sequence $\left(a_n\right)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $\left(d_n(a_n)\right)_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(10 points)

Problem 2. Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii} a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

(10 points)

Problem 3. Let $f(x) = \dfrac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $\left|f^{(n)}(x)\right| < \dfrac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

(10 points)

Problem 4. We say that a subset of $\mathbb{R}^n$ is k-*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers k and n, find the minimal number $d(k, n)$ such that every finite k-generic set in $\mathbb{R}^n$ contains a k-generic subset with at most $d(k, n)$ elements.

(10 points)

Problem 5. For every positive integer n, denote by D_n the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_j \neq j$ for every $1 \le j \le n$. For $1 \le k \le \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \dots, x_n)$ of $(1, 2, \dots, n)$ such that $x_i = k + i$ for every $1 \le i \le k$ and $x_j \neq j$ for every $1 \le j \le n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n - (k+i)}.$$

(10 points)