IMC 2014, Blagoevgrad, Bulgaria

Day 1, July 31, 2014

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\mathrm{trace}(M) = a$ and $\det(M) = b$.

(10 points)

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs (α, β) of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} a_k}{n^\alpha} = \beta$.

(10 points)

Problem 3. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \dots \pm a_1 x \pm a_0$$

has n distinct real roots.

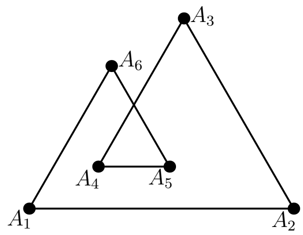
(10 points)

Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorization with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number.

A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

(10 points)

Problem 5. Let $A_1 A_2 \dots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \dots, 3n$, the triangle $A_i A_{i+1} A_{i+2}$ has counterclockwise orientation and $\angle A_i A_{i+1} A_{i+2} = 60°$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$.



(10 points)