IMC 2012, Blagoevgrad, Bulgaria
Day 1, July 28, 2012

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(Proposed by Pavel Kozhevnikov, MIPT, Moscow)

Solution. The transforms given by A and B strictly increase the length of every nonzero vector, this can be seen easily in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product AB also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than -1.

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

(Proposed by Karen Keryan, Yerevan State University, Yerevan, Armenia)

Solution. Let $g(x) = f(x)\cos x$. Since $g(-\pi/2) = g(0) = g(\pi/2) = 0$, by Rolle's theorem there exist some $\xi_1 \in (-\pi/2, 0)$ and $\xi_2 \in (0, \pi/2)$ such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Now consider the function

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x)\cos x - f(x)\sin x}{\cos^2 x}.$$

We have $h(\xi_1) = h(\xi_2) = 0$, so by Rolle's theorem there exist $\xi \in (\xi_1, \xi_2)$ for which

$$0 = h'(\xi) = \frac{g''(\xi)\cos^2\xi + 2\cos\xi\sin\xi g'(\xi)}{\cos^4\xi} =$$

$$= \frac{(f''(\xi)\cos\xi - 2f'(\xi)\sin\xi - f(\xi)\cos\xi)\cos\xi + 2\sin\xi(f'(\xi)\cos\xi - f(\xi)\sin\xi)}{\cos^3\xi} =$$

$$= \frac{f''(\xi)\cos^2\xi - f(\xi)(\cos^2\xi + 2\sin^2\xi)}{\cos^3\xi} = \frac{1}{\cos\xi}(f''(\xi) - f(\xi)(1 + 2\tan^2\xi)).$$

The last yields the desired equality.

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(Proposed by Oleksandr Rybak, Kiev, Ukraine)

Solution. We prove that for any $n \geq 2$ the answer is 6.

First we show that less than 6 trips is not sufficient. In that case the total quantity of students in all trips would not exceed $5n$. A student meets $n-1$ other students in each trip, so he or she takes part on at least 3 excursions to meet all of his or her $2n-1$ schoolmates. Hence the total quantity of students during the trips is not less then $6n$ which is impossible.

Now let's build an example for 6 trips.

If n is even, we may divide $2n$ students into equal groups A, B, C, D. Then we may organize the trips with groups (A, B), (C, D), (A, C), (B, D), (A, D) and (B, C), respectively.

If n is odd and divisible by 3, we may divide all students into equal groups E, F, G, H, I, J. Then the members of trips may be the following: (E, F, G), (E, F, H), (G, H, I), (G, H, J), (E, I, J), (F, I, J).

In the remaining cases let $n = 2x+3y$ be, where x and y are natural numbers. Let's form the groups A, B, C, D of x students each, and E, F, G, H, I, J of y students each. Then we apply the previous cases and organize the following trips: (A, B, E, F, G), (C, D, E, F, H), (A, C, G, H, I), (B, D, G, H, J), (A, D, E, I, J), (B, C, F, I, J).

Problem 4. Let $n \geq 3$ and let $x_1, \ldots, x_n$ be nonnegative real numbers. Define $A = \sum\limits_{i=1}^{n} x_i$, $B = \sum\limits_{i=1}^{n} x_i^2$ and $C = \sum\limits_{i=1}^{n} x_i^3$. Prove that

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$p(X) = \prod_{i=1}^{n}(X - x_i) = X^n - AX^{n-1} + \frac{A^2 - B}{2}X^{n-2} - \frac{A^3 - 3AB + 2C}{6}X^{n-3} + \ldots.$$

The $(n-3)$th derivative of p has three nonnegative real roots $0 \leq u \leq v \leq w$. Hence,

$$\frac{6}{n!}p^{(n-3)}(X) = X^3 - \frac{3A}{n}X^2 + \frac{3(A^2 - B)}{n(n-1)}X - \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)} = (X-u)(X-v)(X-w),$$

so

$$u + v + w = \frac{3A}{n}, \quad uv + vw + wu = \frac{3(A^2 - B)}{n(n-1)} \quad \text{and} \quad uvw = \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)}.$$

From these we can see that

$$\frac{n^2(n-1)^2(n-2)}{9}\big((n+1)A^2B + (n-2)B^2 - A^4 - (2n-2)AC\big) = \ldots =$$

$$= u^2v^2 + v^2w^2 + w^2u^2 - uvw(u+v+w) = uv(u-w)(v-w) + vw(v-u)(w-u) + wu(w-v)(u-v) \geq$$

$$\geq 0 + uw(v-u)(w-v) + wu(w-v)(u-v) = 0.$$

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{\infty} a_n^p$ converges if and only if p is not a prime?

(Proposed by Tomáš Bárta, Charles University, Prague)

Solution. The answer is YES. We prove a more general statement; suppose that $N = C \cup D$ is an arbitrary decomposition of N into two disjoint sets. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n^p$ is convergent for $p \in C$ and divergent for $p \in D$.

Define $C_k = C \cap [1, k]$ and $D_k \cap [1, k]$.

Lemma. For every positive integer k there exists a positive integer N_k and a sequence $X_k = (x_{k,1}, \dots, x_{k,N_k})$ of complex numbers with the following properties:

(a) For $p \in D_k$, we have $\left|\sum_{j=1}^{N_k} x_{k,j}^p\right| \ge 1$.

(b) For $p \in C_k$, we have $\sum_{j=1}^{N_k} x_{k,j}^p = 0$; moreover, $\left|\sum_{j=1}^{m} x_{k,j}^p\right| \le \frac{1}{k}$ holds for $1 \le m \le N_k$.

Proof. First we find some complex numbers $z_1 \dots, z_k$ with

$$\sum_{j=1}^{k} z_j^p = \begin{cases} 0 & p \in C_k \\ 1 & p \in D_k \end{cases} \tag{1}$$

As is well-known, this system of equations is equivalent to another system $\sigma_\nu(z_1, \dots, z_k) = w_\nu$ ($\nu = 1, 2, \dots, k$) where σ_ν is the νth elementary symmetric polynomial, and the constants w_ν are uniquely determined by the Newton-Waring-Girard formulas. Then the numbers $z_1, \dots, z_k$ are the roots of the polynomial $z^k - w_1 z^{k-1} + - \dots + (-1)^k w_k$ in some order.

Now let

$$M = \left\lceil \max_{1 \le m \le k,\ p \in C_k} \left| \sum_{j=1}^{m} z_j^p \right| \right\rceil$$

and let $N_k = k \cdot (kM)^k$. We define the numbers $x_{k,1} \dots, x_{k,N_k}$ by repeating the sequence $(\frac{z_1}{kM}, \frac{z_2}{kM}, \dots, \frac{z_k}{kM})$ $(kM)^k$ times, i.e. $x_{k,\ell} = \frac{z_j}{kM}$ if $\ell \equiv j \pmod{k}$. Then we have

$$\sum_{j=1}^{N_k} x_{k,j}^p = (kM)^k \sum_{j=1}^{k} \left(\tfrac{z_j}{kM}\right)^p = (kM)^{k-p} \sum_{j=1}^{k} z_j^p;$$

then from (1) the properties (a) and the first part of (b) follows immediately. For the second part of (b), suppose that $p \in C_k$ and $1 \le m \le N_k$; then $m = kr + s$ with some integers r and $1 \le s \le k$ and hence

$$\left|\sum_{j=1}^{m} x_{k,j}^p\right| = \left|\sum_{j=1}^{kr} + \sum_{j=kr+1}^{kr+s}\right| = \left|\sum_{j=1}^{s} \left(\frac{z_j}{kM}\right)^p\right| \le \frac{M}{(kM)^p} \le \frac{1}{k}.$$

The lemma is proved.

Now let $S_k = N_1 \dots, N_k$ (we also define $S_0 = 0$). Define the sequence (a) by simply concatenating the sequences $X_1, X_2, \dots$:

$$(a_1, a_2, \dots) = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2}, \dots, x_{k,1}, \dots, x_{k,N_k}, \dots); \tag{1}$$

$$a_{S_k+j} = x_{k+1,j} \quad (1 \le j \le N_{k+1}). \tag{2}$$

If $p \in D$ and $k \ge p$ then

$$\left|\sum_{j=S_k+1}^{S_{k+1}} a_j^p\right| = \left|\sum_{j=1}^{N_{k+1}} x_{k+1,j}^p\right| \ge 1;$$

By Cauchy's convergence criterion it follows that $\sum a_n^p$ is divergent.

If $p \in C$ and $S_u < n \le S_{u+1}$ with some $u \ge p$ then

$$\left|\sum_{j=S_p+1}^{n} a_n^p\right| = \left|\sum_{k=p+1}^{u-1} \sum_{j=1}^{N_k} x_{k,j}^p + \sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| = \left|\sum_{j=1}^{n-S_{u-1}} x_{u,j}^p\right| \le \frac{1}{u}.$$

Then it follows that $\sum_{n=S_p+1}^{\infty} a_n^p = 0$, and thus $\sum_{n=1}^{\infty} a_n^p = 0$ is convergent.