IMC 2013, Blagoevgrad, Bulgaria

Day 1, August 8, 2013

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

(10 points)

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

(10 points)

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}$, $n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(10 points)

Problem 4. Let $n \geq 3$ and let $x_1, \dots, x_n$ be nonnegative real numbers. Define $A = \sum\limits_{i=1}^{n} x_i$, $B = \sum\limits_{i=1}^{n} x_i^2$ and $C = \sum\limits_{i=1}^{n} x_i^3$. Prove that

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC.$$

(10 points)

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{\infty} a_n^p$ converges if and only if p is not a prime?

(10 points)