IMC 2012, Blagoevgrad, Bulgaria

Day 2, July 29, 2012

Problem 1. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}\, x^{2011} + \ldots + a_1\, x + a_0\,.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.

(a) Which of the players has a winning strategy if $m(x) = x - 2012$?

(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

(10 points)

Problem 2. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n} \quad \text{for } n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

(10 points)

Problem 3. Is the set of positive integers n such that $n! + 1$ divides $(2012n)!$ finite or infinite?

(10 points)

Problem 4. Let $n \geq 2$ be an integer. Find all real numbers a such that there exist real numbers $x_1, \ldots, x_n$ satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \ldots = x_{n-1}(1 - x_n) = x_n(1 - x_1) = a.$$

(10 points)

Problem 5. Let $c \geq 1$ be a real number. Let G be an abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y \mid x \in X,\ y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|\underbrace{A + A + \ldots + A}_{k \text{ times}}| \leq c^k |A|$$

for every positive integer k.

(10 points)