IMC 2012, Blagoevgrad, Bulgaria

Day 1, July 28, 2012

Problem 1. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define $p(0) = 1$.

Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

(10 points)

Problem 2. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(10 points)

Problem 3. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \ldots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

(10 points)

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \le 0$ for all $t \ge 0$.

(10 points)

Problem 5. Let a be a rational number and let n be a positive integer. Prove that the polynomial $X^{2^n}(X+a)^{2^n} + 1$ is irreducible in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients.

(10 points)