is a Cauchy sequence in $\mathcal{H}$. (This is the crucial observation.) Indeed, for $m > n$, the norm $\|y_m - y_n\|$ may be computed by the above remark as

$$\|y_m - y_n\|^2 = \frac{d^2}{2}\left\|\left(\frac{1}{m}-\frac{1}{n},\dots,\frac{1}{m}-\frac{1}{n},\frac{1}{m},\dots,\frac{1}{m}\right)^\top\right\|^2_{\mathbb{R}^m} = \frac{d^2}{2}\left(\frac{n(m-n)^2}{m^2n^2}+\frac{m-n}{m^2}\right)$$
$$= \frac{d^2}{2}\frac{(m-n)(m-n+n)}{m^2n} = \frac{d^2}{2}\frac{m-n}{mn} = \frac{d^2}{2}\left(\frac{1}{n}-\frac{1}{m}\right) \to 0, \quad m,n\to\infty.$$

By completeness of $\mathcal{H}$, it follows that there exists a limit

$$y = \lim_{n\to\infty} y_n \in \mathcal{H}.$$

We claim that y sastisfies all conditions of the problem. For $m > n > p$, with n, p fixed, we compute

$$\|x_n - y_m\|^2 = \frac{d^2}{2}\left\|\left(-\frac{1}{m},\dots,-\frac{1}{m},1-\frac{1}{m},-\frac{1}{m},\dots,-\frac{1}{m}\right)^\top\right\|^2_{\mathbb{R}^m}$$
$$= \frac{d^2}{2}\left[\frac{m-1}{m^2}+\frac{(m-1)^2}{m^2}\right] = \frac{d^2}{2}\frac{m-1}{m} \to \frac{d^2}{2}, \quad m\to\infty,$$

showing that $\|x_n - y\| = d/\sqrt{2}$, as well as

$$\langle x_n - y_m, x_p - y_m\rangle = \frac{d^2}{2}\left\langle\left(-\frac{1}{m},\dots,-\frac{1}{m},\dots,1-\frac{1}{m},\dots,-\frac{1}{m}\right)^\top,\right.$$
$$\left.\left(-\frac{1}{m},\dots,1-\frac{1}{m},\dots,-\frac{1}{m},\dots,-\frac{1}{m}\right)^\top\right\rangle_{\mathbb{R}^m}$$
$$= \frac{d^2}{2}\left[\frac{m-2}{m^2}-\frac{2}{m}\left(1-\frac{1}{m}\right)\right] = -\frac{d^2}{2m} \to 0, \quad m\to\infty,$$

showing that $\langle x_n - y, x_p - y\rangle = 0$, so that

$$\left\{\frac{\sqrt{2}}{d}(x_n - y):\ n\in\mathbb{N}\right\}$$

is indeed an orthonormal system of vectors.

This completes the proof in the case when $T = S$, which we can always take if S is countable. If it is not, let x', x'' be any two distinct points in $S\setminus T$. Then applying the above procedure to the set

$$T' = \{x', x'', x_1, x_2, \dots, x_n, \dots\}$$

it follows that

$$\lim_{n\to\infty}\frac{x'+x''+x_1+x_2+\dots+x_n}{n+2} = \lim_{n\to\infty}\frac{x_1+x_2+\dots+x_n}{n} = y$$

satisfies that

$$\left\{\frac{\sqrt{2}}{d}(x'-y), \frac{\sqrt{2}}{d}(x''-y)\right\}\cup\left\{\frac{\sqrt{2}}{d}(x_n - y):\ n\in\mathbb{N}\right\}$$

is still an orthonormal system.

This it true for any distinct $x', x'' \in S\setminus T$; it follows that the entire system

$$\left\{\frac{\sqrt{2}}{d}(x - y):\ x\in S\right\}$$

is an orthonormal system of vectors in $\mathcal{H}$, as required.

IMC2008, Blagoevgrad, Bulgaria

Day 2, July 28, 2008

Problem 1. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

Solution. Let $f(x) = x^{2n} + x^n + 1$, $g(x) = x^{2k} - x^k + 1$, $h(x) = x^{2k} + x^k + 1$. The complex number $x_1 = \cos(\frac{\pi}{3k}) + i\sin(\frac{\pi}{3k})$ is a root of $g(x)$.

Let $\alpha = \frac{\pi n}{3k}$. Since $g(x)$ divides $f(x)$, $f(x_1) = g(x_1) = 0$. So, $0 = x_1^{2n} + x_1^n + 1 = (\cos(2\alpha) + i\sin(2\alpha)) + (\cos\alpha + i\sin\alpha) + 1 = 0$, and $(2\cos\alpha + 1)(\cos\alpha + i\sin\alpha) = 0$. Hence $2\cos\alpha + 1 = 0$, i.e. $\alpha = \pm\frac{2\pi}{3} + 2\pi c$, where $c\in\mathbb{Z}$.

Let x_2 be a root of the polynomial $h(x)$. Since $h(x) = \frac{x^{3k}-1}{x^k-1}$, the roots of the polynomial $h(x)$ are distinct and they are $x_2 = \cos\frac{2\pi s}{3k} + i\sin\frac{2\pi s}{3k}$, where $s = 3a\pm 1, a\in\mathbb{Z}$. It is enough to prove that $f(x_2) = 0$. We have $f(x_2) = x_2^{2n} + x_2^n + 1 = (\cos(4s\alpha) + \sin(4s\alpha)) + (\cos(2s\alpha) + \sin(2s\alpha)) + 1 = (2\cos(2s\alpha) + 1)(\cos(2s\alpha) + i\sin(2s\alpha)) = 0$ (since $2\cos(2s\alpha) + 1 = 2\cos(2s(\pm\frac{2\pi}{3} + 2\pi c)) + 1 = 2\cos(\frac{4\pi s}{3}) + 1 = 2\cos(\frac{4\pi}{3}(3a\pm 1)) + 1 = 0$).

Problem 2. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

Solution. It is well known that an ellipse might be defined by a focus (a point) and a directrix (a straight line), as a locus of points such that the distance to the focus divided by the distance to directrix is equal to a given number $e < 1$. So, if a point X belongs to both ellipses with the same focus F and directrices l_1, l_2, then $e_1\cdot l_1X = FX = e_2\cdot l_2X$ (here we denote by l_1X, l_2X distances between the corresponding line and the point X). The equation $e_1\cdot l_1X = e_2\cdot l_2X$ defines two lines, whose equations are linear combinations with coefficients $e_1, \pm e_2$ of the normalized equations of lines l_1, l_2 but of those two only one is relevant, since X and F should lie on the same side of each directrix. So, we have that all possible points lie on one line. The intersection of a line and an ellipse consists of at most two points.

Problem 3. Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0\le k<n/2}\binom{n}{2k+1}5^k.$$

Solution. As is known, the Fibonacci numbers F_n can be expressed as $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$. Expanding this expression, we obtain that $F_n = \frac{1}{2^{n-1}}\left(\binom{n}{1} + \binom{n}{3}5 + \dots + \binom{n}{l}5^{\frac{l-1}{2}}\right)$, where l is the greatest odd number such that $l\le n$ and $s = \frac{l-1}{2}\le\frac{n}{2}$.

So, $F_n = \frac{1}{2^{n-1}}\sum_{k=0}^{s}\binom{n}{2k+1}5^k$, which implies that 2^{n-1} divides $\sum_{0\le k<n/2}\binom{n}{2k+1}5^k$.

Problem 4. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients, and let $f(x), g(x)\in\mathbb{Z}[x]$ be nonconstant polynomials such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if the polynomial $f(x)-2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is greater than 5.

Solution. Let $f(x) = g(x)h(x)$ where $h(x)$ is a polynomial with integer coefficients.

Let $a_1, \dots, a_{81}$ be distinct integer roots of the polynomial $f(x) - 2008$. Then $f(a_i) = g(a_i)h(a_i) = 2008$ for $i = 1, \dots, 81$, Hence, $g(a_1), \dots, g(a_{81})$ are integer divisors of 2008.

Since $2008 = 2^3\cdot 251$ $(2, 251$ are primes) then 2008 has exactly 16 distinct integer divisors (including the negative divisors as well). By the pigeonhole principle, there are at least 6 equal numbers among $g(a_1), \dots, g(a_{81})$ (because $81 > 16\cdot 5$). For example, $g(a_1) = g(a_2) = \dots = g(a_6) = c$. So $g(x) - c$ is a nonconstant polynomial which has at least 6 distinct roots (namely $a_1, \dots, a_6$). Then the degree of the polynomial $g(x) - c$ is at least 6.

Problem 5. Let n be a positive integer, and consider the matrix $A = (a_{ij})_{1 \le i,j \le n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $|\det A| = k^2$ for some integer k.

Solution. Call a square matrix *of type (B)*, if it is of the form

$$\begin{pmatrix} 0 & b_{12} & 0 & \dots & b_{1,2k-2} & 0 \\ b_{21} & 0 & b_{23} & \dots & 0 & b_{2,2k-1} \\ 0 & b_{32} & 0 & \dots & b_{3,2k-2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2k-2,1} & 0 & b_{2k-2,3} & \dots & 0 & b_{2k-2,2k-1} \\ 0 & b_{2k-1,2} & 0 & \dots & b_{2k-1,2k-2} & 0 \end{pmatrix}.$$

Note that every matrix of this form has determinant zero, because it has k columns spanning a vector space of dimension at most $k-1$.

Call a square matrix *of type (C)*, if it is of the form

$$C' = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & \dots & 0 & c_{1,k} \\ c_{11} & 0 & c_{12} & 0 & \dots & c_{1,k} & 0 \\ 0 & c_{21} & 0 & c_{22} & \dots & 0 & c_{2,k} \\ c_{21} & 0 & c_{22} & 0 & \dots & c_{2,k} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{k,1} & 0 & c_{k,2} & \dots & 0 & c_{k,k} \\ c_{k,1} & 0 & c_{k,2} & 0 & \dots & c_{k,k} & 0 \end{pmatrix}$$

By permutations of rows and columns, we see that

$$|\det C'| = \left| \det \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right| = |\det C|^2,$$

where C denotes the $k \times k$-matrix with coefficients $c_{i,j}$. Therefore, the determinant of any matrix of type (C) is a perfect square (up to a sign).

Now let X' be the matrix obtained from A by replacing the first row by $\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$, and let Y be the matrix obtained from A by replacing the entry a_{11} by 0. By multi-linearity of the determinant, $\det(A) = \det(X') + \det(Y)$. Note that X' can be written as

$$X' = \begin{pmatrix} 1 & 0 \\ v & X \end{pmatrix}$$

for some $(n-1) \times (n-1)$-matrix X and some column vector v. Then $\det(A) = \det(X) + \det(Y)$. Now consider two cases. If n is odd, then X is of type (C), and Y is of type (B). Therefore, $|\det(A)| = |\det(X)|$ is a perfect square. If n is even, then X is of type (B), and Y is of type (C); hence $|\det(A)| = |\det(Y)|$ is a perfect square.

The set of primes can be replaced by any subset of $\{2\} \cup \{3, 5, 7, 9, 11, \dots\}$.

Problem 6. Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in $\mathcal{H}$ such that the distance between any two distinct points in S is equal to d. Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : \ x \in S \right\}$$

is an orthonormal system of vectors in $\mathcal{H}$.

Solution. It is clear that, if $\mathcal{B}$ is an orthonormal system in a Hilbert space $\mathcal{H}$, then $\{(d/\sqrt{2})e : \ e \in \mathcal{B}\}$ is a set of points in $\mathcal{H}$, any two of which are at distance d apart. We need to show that every set S of equidistant points is a translate of such a set.

We begin by noting that, if $x_1, x_2, x_3, x_4 \in S$ are four distinct points, then

$$\begin{aligned} \langle x_2 - x_1, x_2 - x_1 \rangle &= d^2, \\ \langle x_2 - x_1, x_3 - x_1 \rangle &= \frac{1}{2}\left(\|x_2 - x_1\|^2 + \|x_3 - x_1\|^2 - \|x_2 - x_3\|^2 \right) = \frac{1}{2} d^2, \\ \langle x_2 - x_1, x_4 - x_3 \rangle &= \langle x_2 - x_1, x_4 - x_1 \rangle - \langle x_2 - x_1, x_3 - x_1 \rangle = \frac{1}{2} d^2 - \frac{1}{2} d^2 = 0. \end{aligned}$$

This shows that scalar products among vectors which are finite linear combinations of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n,$$

where $x_1, x_2, \dots, x_n$ are distinct points in S and $\lambda_1, \lambda_2, \dots, \lambda_n$ are integers with $\lambda_1 + \lambda_2 + \dots + \lambda_n = 0$, are universal across all such sets S in all Hilbert spaces $\mathcal{H}$; in particular, we may conveniently evaluate them using examples of our choosing, such as the canonical example above in $\mathbb{R}^n$. In fact this property trivially follows also when coefficients λ_i are rational, and hence by continuity any real numbers with sum 0.

If $S = \{x_1, x_2, \dots, x_n\}$ is a finite set, we form

$$x = \frac{1}{n}\left(x_1 + x_2 + \dots + x_n \right),$$

pick a non-zero vector $z \in [\operatorname{Span}(x_1 - x, x_2 - x, \dots, x_n - x)]^\perp$ and seek y in the form $y = x + \lambda z$ for a suitable $\lambda \in \mathbb{R}$. We find that

$$\langle x_1 - y, x_2 - y \rangle = \langle x_1 - x - \lambda z, x_2 - x - \lambda z \rangle = \langle x_1 - x, x_2 - x \rangle + \lambda^2 \|z\|^2.$$

$\langle x_1 - x, x_2 - x \rangle$ may be computed by our remark above as

$$\begin{aligned} \langle x_1 - x, x_2 - x \rangle &= \frac{d^2}{2} \left\langle \left(\frac{1}{n} - 1, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top, \left(\frac{1}{n}, \frac{1}{n} - 1, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top \right\rangle_{\mathbb{R}^n} \\ &= \frac{d^2}{2} \left(\frac{2}{n}\left(\frac{1}{n} - 1 \right) + \frac{n-2}{n^2} \right) = -\frac{d^2}{2n}. \end{aligned}$$

So the choice $\lambda = \dfrac{d}{\sqrt{2n}\|z\|}$ will make all vectors $\dfrac{\sqrt{2}}{d}(x_i - y)$ orthogonal to each other; it is easily checked as above that they will also be of length one.

Let now S be an infinite set. Pick an infinite sequence $T = \{x_1, x_2, \dots, x_n, \dots\}$ of distinct points in S. We claim that the sequence

$$y_n = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$$