

**5<sup>th</sup> INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS**

**July 29 - August 3, 1998, Blagoevgrad, Bulgaria**

**First day**

*PROBLEMS AND SOLUTIONS*

**Problem 1.** (20 points) Let  $V$  be a 10-dimensional real vector space and  $U_1$  and  $U_2$  two linear subspaces such that  $U_1 \subseteq U_2$ ,  $\dim_{\mathbb{R}} U_1 = 3$  and  $\dim_{\mathbb{R}} U_2 = 6$ . Let  $\mathcal{E}$  be the set of all linear maps  $T : V \rightarrow V$  which have  $U_1$  and  $U_2$  as invariant subspaces (i.e.,  $T(U_1) \subseteq U_1$  and  $T(U_2) \subseteq U_2$ ). Calculate the dimension of  $\mathcal{E}$  as a real vector space.

**Solution** First choose a basis  $\{v_1, v_2, v_3\}$  of  $U_1$ . It is possible to extend this basis with vectors  $v_4, v_5$  and  $v_6$  to get a basis of  $U_2$ . In the same way we can extend a basis of  $U_2$  with vectors  $v_7, \dots, v_{10}$  to get as basis of  $V$ .

Let  $T \in \mathcal{E}$  be an endomorphism which has  $U_1$  and  $U_2$  as invariant subspaces. Then its matrix, relative to the basis  $\{v_1, \dots, v_{10}\}$  is of the form

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

So  $\dim_{\mathbb{R}} \mathcal{E} = 9 + 18 + 40 = 67$ .

**Problem 2.** Prove that the following proposition holds for  $n = 3$  (5 points) and  $n = 5$  (7 points), and does not hold for  $n = 4$  (8 points).

“For any permutation  $\pi_1$  of  $\{1, 2, \dots, n\}$  different from the identity there is a permutation  $\pi_2$  such that any permutation  $\pi$  can be obtained from  $\pi_1$  and  $\pi_2$  using only compositions (for example,  $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$ ).”

**Solution**

Let  $S_n$  be the group of permutations of  $\{1, 2, \dots, n\}$ .

1) When  $n = 3$  the proposition is obvious: if  $x = (12)$  we choose  $y = (123)$ ; if  $x = (123)$  we choose  $y = (12)$ .

2)  $n = 4$ . Let  $x = (12)(34)$ . Assume that there exists  $y \in S_n$ , such that  $S_4 = \langle x, y \rangle$ . Denote by  $K$  the invariant subgroup

$$K = \{id, (12)(34), (13)(24), (14)(23)\}.$$

By the fact that  $x$  and  $y$  generate the whole group  $S_4$ , it follows that the factor group  $S_4/K$  contains only powers of  $\bar{y} = yK$ , i.e.,  $S_4/K$  is cyclic. It is easy to see that this factor-group is not commutative (something more this group is not isomorphic to  $S_3$ ).

3)  $n = 5$

a) If  $x = (12)$ , then for  $y$  we can take  $y = (12345)$ .

b) If  $x = (123)$ , we set  $y = (124)(35)$ . Then  $y^3xy^3 = (125)$  and  $y^4 = (124)$ . Therefore  $(123), (124), (125) \in \langle x, y \rangle$ - the subgroup generated by  $x$  and  $y$ . From the fact that  $(123), (124), (125)$  generate the alternating subgroup  $A_5$ , it follows that  $A_5 \subset \langle x, y \rangle$ . Moreover  $y$  is an odd permutation, hence  $\langle x, y \rangle = S_5$ .

c) If  $x = (123)(45)$ , then as in b) we see that for  $y$  we can take the element  $(124)$ .

d) If  $x = (1234)$ , we set  $y = (12345)$ . Then  $(yx)^3 = (24) \in \langle x, y \rangle$ ,  $x^2(24) = (13) \in \langle x, y \rangle$  and  $y^2 = (13524) \in \langle x, y \rangle$ . By the fact  $(13) \in \langle x, y \rangle$  and  $(13524) \in \langle x, y \rangle$ , it follows that  $\langle x, y \rangle = S_5$ .

e) If  $x = (12)(34)$ , then for  $y$  we can take  $y = (1354)$ . Then  $y^2x = (125)$ ,  $y^3x = (124)(53)$  and by c)  $S_5 = \langle x, y \rangle$ .

f) If  $x = (12345)$ , then it is clear that for  $y$  we can take the element  $y = (12)$ .

**Problem 3.** Let  $f(x) = 2x(1 - x)$ ,  $x \in \mathbb{R}$ . Define

$$f_n = \overbrace{f \circ \dots \circ f}^n.$$

a) (10 points) Find  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ .

b) (10 points) Compute  $\int_0^1 f_n(x) dx$  for  $n = 1, 2, \dots$

**Solution.** a) Fix  $x = x_0 \in (0, 1)$ . If we denote  $x_n = f_n(x_0)$ ,  $n = 1, 2, \dots$  it is easy to see that  $x_1 \in (0, 1/2]$ ,  $x_1 \leq f(x_1) \leq 1/2$  and  $x_n \leq f(x_n) \leq 1/2$  (by induction). Then  $(x_n)_n$  is a bounded non-decreasing sequence and, since  $x_{n+1} = 2x_n(1 - x_n)$ , the limit  $l = \lim_{n \rightarrow \infty} x_n$  satisfies  $l = 2l(1 - l)$ , which implies  $l = 1/2$ . Now the monotone convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 1/2.$$

b) We prove by induction that

$$(1) \quad f_n(x) = \frac{1}{2} - 2^{2^n - 1} \left( x - \frac{1}{2} \right)^{2^n}$$

holds for  $n = 1, 2, \dots$ . For  $n = 1$  this is true, since  $f(x) = 2x(1 - x) = \frac{1}{2} - 2(x - \frac{1}{2})^2$ . If (1) holds for some  $n = k$ , then we have

$$\begin{aligned} f_{k+1}(x) &= f_k(f(x)) = \frac{1}{2} - 2^{2^k - 1} \left( \left( \frac{1}{2} - 2(x - \frac{1}{2})^2 \right) - \frac{1}{2} \right)^{2^k} \\ &= \frac{1}{2} - 2^{2^k - 1} \left( -2(x - \frac{1}{2})^2 \right)^{2^k} \\ &= \frac{1}{2} - 2^{2^{k+1} - 1} \left( x - \frac{1}{2} \right)^{2^{k+1}} \end{aligned}$$

which is (2) for  $n = k + 1$ .

Using (1) we can compute the integral,

$$\int_0^1 f_n(x) dx = \left[ \frac{1}{2}x - \frac{2^{2^n - 1}}{2^n + 1} \left( x - \frac{1}{2} \right)^{2^n + 1} \right]_{x=0}^1 = \frac{1}{2} - \frac{1}{2(2^n + 1)}.$$

**Problem 4.** (20 points) The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable and satisfies  $f(0) = 2$ ,  $f'(0) = -2$  and  $f(1) = 1$ . Prove that there exists a real number  $\xi \in (0, 1)$  for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

**Solution.** Define the function

$$g(x) = \frac{1}{2}f^2(x) + f'(x).$$

Because  $g(0) = 0$  and

$$f(x) \cdot f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number  $0 < \eta \leq 1$  for which  $g(\eta) = 0$ .

a) If  $f$  is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because  $h(0) = h(1) = -\frac{1}{2}$ , there exists a real number  $0 < \eta < 1$  for which  $h'(\eta) = 0$ . But  $g = f^2 \cdot h'$ , and we are done.

b) If  $f$  has at least one zero, let  $z_1$  be the first one and  $z_2$  be the last one. (The set of the zeros is closed.) By the conditions,  $0 < z_1 \leq z_2 < 1$ .

The function  $f$  is positive on the intervals  $[0, z_1)$  and  $(z_2, 1]$ ; this implies that  $f'(z_1) \leq 0$  and  $f'(z_2) \geq 0$ .

Then  $g(z_1) = f'(z_1) \leq 0$  and  $g(z_2) = f'(z_2) \geq 0$ , and there exists a real number  $\eta \in [z_1, z_2]$  for which  $g(\eta) = 0$ .

**Remark.** For the function  $f(x) = \frac{2}{x+1}$  the conditions hold and  $f \cdot f' + f''$  is constantly 0.

**Problem 5.** Let  $P$  be an algebraic polynomial of degree  $n$  having only real zeros and real coefficients.

a) (15 points) Prove that for every real  $x$  the following inequality holds:

$$(2) \quad (n-1)(P'(x))^2 \geq nP(x)P''(x).$$

b) (5 points) Examine the cases of equality.

**Solution.** Observe that both sides of (2) are identically equal to zero if  $n = 1$ . Suppose that  $n > 1$ . Let  $x_1, \dots, x_n$  be the zeros of  $P$ . Clearly (2) is true when  $x = x_i$ ,  $i \in \{1, \dots, n\}$ , and equality is possible only if  $P'(x_i) = 0$ , i.e., if  $x_i$  is a multiple zero of  $P$ . Now suppose that  $x$  is not a zero of  $P$ . Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^n \frac{1}{x-x_i}, \quad \frac{P''(x)}{P(x)} = \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_i)(x-x_j)},$$

we find

$$(n-1) \left( \frac{P'(x)}{P(x)} \right)^2 - n \frac{P''(x)}{P(x)} = \sum_{i=1}^n \frac{n-1}{(x-x_i)^2} - \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_i)(x-x_j)}.$$

But this last expression is simply

$$\sum_{1 \leq i < j \leq n} \left( \frac{1}{x-x_i} - \frac{1}{x-x_j} \right)^2,$$

and therefore is positive. The inequality is proved. In order that (2) holds with equality sign for every real  $x$  it is necessary that  $x_1 = x_2 = \dots = x_n$ . A direct verification shows that indeed, if  $P(x) = c(x-x_1)^n$ , then (2) becomes an identity.

**Problem 6.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function with the property that for any  $x$  and  $y$  in the interval,

$$xf(y) + yf(x) \leq 1.$$

a) (15 points) Show that

$$\int_0^1 f(x) dx \leq \frac{\pi}{4}.$$

b) (5 points) Find a function, satisfying the condition, for which there is equality.

**Solution** Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin \theta) \cos \theta d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta d\theta$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}.$$

Now let  $f(x) = \sqrt{1-x^2}$ . If  $x = \sin \theta$  and  $y = \sin \phi$  then

$$xf(y) + yf(x) = \sin \theta \cos \phi + \sin \phi \cos \theta = \sin(\theta + \phi) \leq 1.$$